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"Solitary Waves in Channels"

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Solitary Waves in Channels

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I. Introduction

In these lectures we will focus our attention on a solitary surface wave traveling in one direction along a channel. The wavelength will be much larger than the depth of the channel, therefore a shallow water theory will here be applied in developing the Korteweg-de Vries (KdV) equation. The nonlinear effects of short waves (Stokes waves) will not be described here. However, the reader will have the basic equations from which the nonlinear corrections for waves of small amplitude (with respect to wavelength) on deep water can also be obtained. Eagleson *al.* (1966) gave a simple view of the ranges of different wave theories. To gain a better understanding of dynamics of a solitary wave another approach, which is used more often in nonlinear problems of hydraulics will also be followed. The lecture will finish with a brief discussion of internal solitary waves.

II. Governing equations

The law of conservation of mass of a fluid is the equation of continuity:

$$\rho_t + \nabla \bullet (\rho \boldsymbol{u}) = 0.$$

We will suppose an incompressible fluid in which the density ρ is constant.Therefore the equation of continuity is replaced by the equation for volume conservation (incompressible fluid):

$$\nabla \bullet \boldsymbol{u} = 0. \tag{2.1}$$

We will also suppose that the effect of friction is negligible. The only body force acting on a fluid is the gravity. We shall presume that the time-scale of motion (the period) is much shorter than the period of Earth rotation, so that the Coriolis force will also be ignored. The motion of the fluid is therefore described by the Euler equation:

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot)\boldsymbol{u} = -\frac{\nabla p}{\rho} - \boldsymbol{k}g, \tag{2.2}$$

In (2.1) and (2.2) the velocity $\boldsymbol{u}$ and pressure p are the unknown variables, g is the gravity acceleration and $\boldsymbol{k}$ is the unit vector in positive (upward) z direction. The $\nabla\times$ operator on (2.2) gives equation for vorticity $\boldsymbol{\omega}$

$$\boldsymbol{\omega}_t + (\boldsymbol{u}\cdot)\boldsymbol{\omega} \equiv \frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot)\boldsymbol{u},$$

which means that the vorticity is conserved following the fluid particle, if its initial vorticity was zero. Before the fluid was disturbed its velocity was zero. After the initialization of disturbance the vorticity will remain zero far from the boundary layers around the body which generated it. We shall take the motion to be irrotational

$$\boldsymbol{\omega} = \boldsymbol{0}, \tag{2.3}$$

as it is in many problems of water waves. This means that the velocity potential ϕ may be introduced

$$\boldsymbol{u} = \nabla\phi \tag{2.4}$$

and the continuity equation (2.1) transforms into the Laplace equation for the potential

$$\nabla^2\phi = 0. \tag{2.5}$$

The advective term in Euler equation (2.2) can be expressed with the vorticity through the vector identity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\boldsymbol{\omega}\times\boldsymbol{v}+\nabla\cdot(\boldsymbol{u}\cdot\boldsymbol{u})$. When we introduce the potential with (2.4), the equation of motion becomes

$$\nabla\left[\phi_t+\frac{(\nabla\phi)^2}{2}+\frac{p}{\rho}+gz\right]=0.$$

Integration of this equation with respect to the space variables gives

$$\phi_t+\frac{(\nabla\phi)^2}{2}+\frac{p}{\rho}+gz=C(t).$$

The constant C is a function of time and is determined by the pressure imposed at the boundary (the surface) of the fluid in motion. Any function of time alone may be added to the potential ϕ to have the same velocity field, or we may take $C(t) = 0$ without loss of generality. Therefore:

$$\phi_t+\frac{(\nabla\phi)^2}{2}+\frac{p}{\rho}+gz=0. \qquad (2.6)$$

The Bernoulli equation (2.6) and Laplace equation (2.5) describe the velocity and pressure field for unsteady irrotational motions in the fluid.

III. Boundary conditions

Boundary conditions represent a problem in theories of waves of finite amplitude since they are also imposed on an unknown free surface of fluid. We shall begin with the kinematic boundary condition at the free surface, which has an elevation $\eta(x, t)$ (Fig. 1) above the mean-zero level, varying along the channel axis x. The particle at the free surface will remain on it (unless the wave breaks), meaning that if we follow the particle at the surface, its vertical displacement z = η will remain unchanged:

$$\frac{D(z-\eta)}{Dt}=0.$$

Since the flow is irrotational, we replace the velocity components (*u*, *w*) in the above expression with the potential at the free surface. Since D*z*/D*t* = *w*, we obtain for the kinematic condition at the free surface:

$$\eta_t+\phi_x\eta_x-\phi_z=0; \quad z=\eta. \tag{3.1}$$

The Bernoulli equation (2.6) holds also at the fluid surface, where we shall ignore the effect of surface tension (waves are supposed to have a wavelength larger than 0.1 m). The pressure just below the surface is then equal to the atmospheric pressure above it. The atmospheric pressure over the fluid surface is assumed to be constant (zero). The *dynamic* boundary condition therefore yields

$$\phi_t+\frac{(\nabla\phi)^2}{2}+\frac{p}{\rho}+g\eta=0; \quad z=\eta. \tag{3.2}$$

At the channel floor (z = -$H(x)$) the horizontal velocity component does not vanish in inviscid fluid, and the advected fluid particles near the sloping bottom also have a vertical velocity component. The boundary condition at the channel bottom is similar to the condition (3.1):

$$\phi_x H_x+\phi_z=0; \quad z=-H(x). \tag{3.3}$$

Over the flat bottom the condition simplifies into the form

$$\phi_z=0; \quad z=-H. \tag{3.4}$$

IV. Linear wave theory

We shall examine when the nonlinear boundary conditions (3.1) and (3.2) at the free surface could be linearized. Let a be the amplitude of a wave and τ the period in which the fluid particles travel a distance of the order a in a wave of wavelength λ. The water is supposed to be deep enough, therefore the velocity scale of fluid particles u is of the order a/τ. If the advective term in Euler equation (2.2) is much smaller than the local acceleration term, then:

$$\frac{a^2}{\lambda\tau^2} \ll \frac{a}{\tau^2},$$

or

$$a \ll \lambda. \tag{4.1}$$

The slope of the surface, which is of the order $a/\lambda << 1$, is gentle. Since $\lambda = c\tau$, where c is the phase speed of wave propagation, and $a \cong u\tau$, the condition (4.1) also means that $u << c$, or

$$F \ll 1, \tag{4.2}$$

where $F = u/c$ is the Froude number, which plays an important role in determination of hydraulic regimes of the flow in a channel. The condition (4.1) is sufficient for the linearization of boundary conditions (3.1) and (3.2), which now take the form:

$$\eta_t - \phi_z = 0; \quad z = \eta \tag{4.3}$$

and

$$\phi_t + g\eta = 0; \quad z = \eta. \tag{4.4}$$

This, however, is not enough. We still have the boundary conditions at the unknown free surface. On expanding the derivatives ϕ_t and ϕ_z in the Taylor series around $z = 0$ it becomes evident that the derivatives of ϕ could be taken at the level of undisturbed free surface $z = 0$, when $a << \lambda$. The surface boundary conditions now follow as

$$\eta_t - \phi_z = 0; \quad z = 0 \tag{4.5}$$
$$\phi_t + g\eta = 0; \quad z = 0. \tag{4.6}$$

A condition different from (4.1) becomes important for the linearization of the problem if we are concerned only with long waves traveling in a shallow channel, when $H << \lambda$. Now the vertical velocity w is estimated again with a/τ, while the estimate of the horizontal velocity component follows from the continuity equation (2.1), which yields $u \cong (a\lambda)/(\tau H)$. The condition of linearization (4.1) is replaced with the condition

$$a \ll H, \tag{4.7}$$

which is also sufficient for the replacements of derivatives of ϕ at the free surface with the ones at fixed level $z = 0$.

At the fiat channel bottom the condition remains unchanged

$$\phi_z = 0; \quad z = -H. \tag{4.8}$$

The solution of Laplace equation for potential

$$\nabla^2 \phi = 0 \tag{4.9}$$

with boundary conditions (4.5), (4.6) and (4.8) can be found in textbooks of fluid dynamics, like Paterson (1983) and Acheson (1990) . We shall just mention that when the surface is supposed to be in a form of a traveling wave

$$\eta = a\cos\left(kx - \omega t\right) \tag{4.10}$$

the velocity potential is obtained from (4.9) by considering the kinematic boundary conditions at the channel bottom (4.8), and at the free surface(4.5):

$$\phi = \frac{a\omega}{k}\frac{\cosh k\left(z+H\right)}{\sinh\left(kH\right)}\sin\left(kx-\omega t\right), \tag{4.11}$$

$$u = a\omega\frac{\cosh k\left(z+H\right)}{\sinh\left(kH\right)}\cos\left(kx-\omega t\right), \tag{4.12}$$

$$w = a\omega\frac{\sinh k\left(z+H\right)}{\sinh\left(kH\right)}\sin\left(kx-\omega t\right). \tag{4.13}$$

Finally, the dynamic boundary condition (4.6) gives us the dispersion relation:

$$\omega = \sqrt{gk\tanh\left(kH\right)}. \tag{4.14}$$

The phase velocity

$$c = \sqrt{\frac{g}{k}\tanh\left(kH\right)} \tag{4.15}$$

differs from the group velocity

$$c_g = \sqrt{\frac{g}{k}}\frac{\left[\sinh\left(kH\right)\cosh\left(kh\right)+kH\right]}{2\cosh^2\left(kH\right)\sqrt{\tanh\left(kH\right)}} = \frac{c}{2}\left[1+\frac{2k(\omega)H}{\sinh\left(2k(\omega)H\right)}\right]. \tag{4.16}$$

The linear theory covers surface waves on deep water as well as on shallow water. For deep water ($H >> \lambda$.) the amplitude of the vertical velocity component is equal to the amplitude of the horizontal velocity component, they both decrease exponentialy with depth (like e^{kz}; Fig. 2). The fluid particle paths are then almost circular (Stokes' drift is still present). The dispersion (c = $2c_g$ = $(g/k)^{0.5}$) spreads the packet of these short waves.

We shall be more oriented towards long waves, when $H << \lambda$ or when $kH << 1$. The linear long waves are nondispersive

$$c = c_g = \sqrt{gH}. \tag{4.17}$$

The velocity potential is approximated with

$$\phi = \frac{a\omega}{k^2 H}\sin(kx - \omega t), \tag{4.18}$$

The amplitude of the horizontal velocity component does not decrease with depth, so

$$u = \frac{a\omega}{kH}\cos(kx - \omega t), \tag{4.19}$$

while the amplitude of the vertical velocity component decreases linearly towards the bottom

$$w = a\omega\left(1 + \frac{z}{H}\right)\sin(kx - \omega t). \tag{4.20}$$

It could also be shown from the linearized Euler equation (2.2) that the pressure changes with depth hydrostatically.

At the end the linear wave equation has to be written. From the kinematic boundary condition at the free surface (4.5) it follows:

$$\eta_t + Hu_x = 0, \tag{4.21}$$

where the vertical velocity component ϕ_z (=w) was expressed with the horizontal velocity component by the integration of (2.1) with respect to z from $z = -H$ to $z = 0$ (u_x is depth independent). We derivate the dynamic boundary condition at the surface (4.6) with respect to x to obtain

$$u_t + g\eta_x = 0 \tag{4.22}$$

Both boundary conditions at the free surface, (4.21) and (4.22) give the wave equations:

$$u_{tt} - c^2 u_{xx} = 0; \quad u_{tt} - c^2 u_{xx} = 0. \tag{4.23}$$

V. Nonlinear shallow water equations

In the previous section the following ratios were introduced:

$$\varepsilon = \frac{a}{H}, \quad \delta = \frac{H^2}{L^2}, \tag{5.1}$$

where a is the wave amplitude, $L(=\lambda)$ the horizontal length scale, and H the vertical length scale. The ratios must be small for the linearized theory to be applied . In order to develop the nonlinear shallow water equations, it is convenient to introduce the nondimensional flow variables:

$$x^* = \frac{x}{L}, \quad z^* = \frac{z}{H}, \quad \eta^* = \frac{\eta}{a}, \quad t^* = \frac{ct}{L} \tag{5.2}$$

where c (=$(gH)^{0.5}$) is the phase speed on a shallow water (4.17). By inserting (5.2) in dynamic boundary condition (3.2) at the free surface we obtain the dimensionless velocity potential ϕ^* as:

$$\phi^* = \frac{H}{caL}\phi. \tag{5.3}$$

The basic equations for water waves in a channel with a flat bottom (2.5), (3.1), (3.2), and (3.4) can be expressed in the nondimensional form:

$$\delta\phi_{xx} + \phi_{zz} = 0, \tag{5.4}$$

$$\delta\left[\eta_t + \varepsilon\eta_x\phi_x\right] - \phi_z = 0, \quad z = \varepsilon\eta, \tag{5.5}$$

$$\phi_t + \frac{\varepsilon}{2}\phi_x^2 + \frac{\varepsilon}{2\delta}\phi_z^2 + \eta = 0; \quad z = \varepsilon\eta, \tag{5.6}$$

$$\phi_z = 0, \quad z = -1, \tag{5.7}$$

where the asterisks were dropped for simplicity. At this step we could also be interested in short waves, for which $\delta \geq 1$. Then we would look for the solution in the form of a series of ε, which would lead to the Stokes wave. Instead, we follow Debnath (1994) and seek for a solution of (5.4)-(5.7) for shallow water ($\delta << 1$) in a form

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + ... \tag{5.8}$$

Substitution of ϕ from (5.8) in the Laplace equation (5.4) gives to zero order

$$\phi_{0zz} = 0. \tag{5.9}$$

We integrate (5.9) with respect to z from the bottom z = -1, where the boundary condition (5.7) holds. This implies the integration constant zero, and to zero order the vertical velocity $\phi_{0z} = 0$ for every $z \in (-1, \varepsilon\delta)$. Therefore, the potential $\phi_0(x, t)$ is independent of the vertical coordinate z. We shall denote

$$\phi_{0x} = u(x,t). \tag{5.10}$$

Laplace equation (5.4) to first and second order is

$$\phi_{1zz} = -\phi_{0xx} = -u_x, \tag{5.11}$$

$$\phi_{2zz} = -\phi_{1xx}, \tag{5.12}$$

from which upon two subsequent integrations with respect to z and consideration of (5.7) we obtain first-order and second-order corrections to ϕ

$$\phi_1 = -\frac{u_x}{2}(z+1)^2, \tag{5.13}$$

$$\phi_2 = \frac{u_{xxx}}{24}(z+1)^4. \tag{5.14}$$

First and second order terms reveal that the series (5.8) is a power series of ϕ in even powers of (z + 1) about its value at the bottom ϕ_0:

$$\phi = \phi_0 - \delta\frac{u_x(z+1)^2}{2} + \delta^2\frac{u_{xxx}(z+1)^4}{2} + \ldots \tag{5.15}$$

This series makes sense since for shallow water waves the orbits of fluid particles are horizontaly elongated and the vertical velocity component ϕ_z is small.

We will suppose now that δ and ε are of the same order of magnitude. The kinematic boundary condition (5.5) at the surface can be written in a form which includes terms up to order $\delta\varepsilon$, δ^2 and ε^2

$$\delta\left[\eta_t + \varepsilon u\eta_x + (1+\varepsilon\eta)u_x\right] = \frac{\delta^2}{6}u_{xxx} \tag{5.16}$$

and the dynamic boundary condition by retaining terms up to order ε and δ

$$\phi_{0t} - \frac{\delta}{2}u_{xt} + \frac{\varepsilon}{2}u^2 + \eta = 0. \tag{5.17}$$

We derivate (5.16) with respect to x and simplify (5.17) to obtain

$$\eta_t + \left[(1+\varepsilon\eta)u\right]_x = \frac{\delta}{6}u_{xxx} \tag{5.18}$$

$$u_t + \varepsilon u u_x + \eta_x - \frac{\delta}{2}u_{txx} = 0. \tag{5.19}$$

The pair of equations (5.18)-(5.19) is accurate to the second order and is known as the Boussinesq system of equations for shallow water. By ignoring terms with

ε and δ we get the system of zero order:

$$u_t + \eta_x = 0,$$
$$\eta_t + u_x = 0$$

which leads to the dimensionless wave equations

$$u_{tt} - u_{xx} = 0; \quad \eta_{tt} - \eta_{xx} = 0, \tag{5.20}$$

with dimensionless phase speed c = l. Their dimensional form is (4.23). Eqs. (5.20) describe the long wave without any dispersion.

In zero order u is linearily proportional to η, as follows from (5.20). Consequently, we seek for the solution of (5.18)-(5.19) in a form

$$u = \eta + \varepsilon P + \delta Q, \tag{5.21}$$

where P and Q are unknown functions of η. This transforms the system (5.18)-(5.19) up to order δ and ε

$$\eta_t + \eta_x + \varepsilon\left(P_x + 2\eta\eta_x\right) + \delta\left(Q_x - \frac{\eta_{xxx}}{6}\right) = 0, \tag{5.22}$$

$$\eta_t + \eta_x + \varepsilon\left(P_t + \eta\eta_x\right) + \delta\left(Q_t - \frac{\eta_{txx}}{2}\right) = 0. \tag{5.23}$$

For the zero order

$$\eta_t = -\eta_x \tag{5.24}$$

Subtraction of (5.23) from (5.22) gives

$$\varepsilon\left(P_t - P_x - \eta\eta_x\right) + \delta\left(Q_t - Q_x - \frac{\eta_{txx}}{2} + \frac{\eta_{xxx}}{6}\right) = 0. \tag{5.25}$$

Parameters δ and ε are supposed to be of the same order, but they are not equal. Since (5.22) and (5.23) should represent the same equation up to first order of δ and ε, the terms in parentheses have to vanish[1]. This happens if

$$P = -\frac{\eta^2}{4}, \quad Q = \frac{\eta_{xx}}{3} \tag{5.26}$$

The velocity is then expressed with the elevation as:

$$u = \eta - \varepsilon\frac{\eta^2}{4} + \delta\frac{\eta_{xx}}{4} \tag{5.27}$$

and (5.22), or (5.23) becomes

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x + \frac{1}{6}\delta\eta_{xxx} = 0. \tag{5.28}$$

This is the *Korteweg-de Vries* (*KdV*) equation, named after the authors who derived it in 1895. It was obtained with the assumption that $\varepsilon/\delta = aL^2/H^3 \sim 1$. The dimensional equivalent of KdV is:

$$\eta_t + c\left(1 + \frac{3}{2H}\eta\right)\eta_x + \frac{cH^2}{6}\eta_{xxx} = 0. \tag{5.29}$$

The first two terms $\eta_t + c\eta_x$ represent the wave evolution at the speed $c = (gH)^{1/2}$ in shallow water. The third, nonlinear term, is responsible for a wave steepening (change in amplitude), while the fourth, linear term describes the frequency dispersion. The balance between these effects gives also a solitary wave. If the undisturbed depth of the channel H is increased, the nonlinear steepening term is reduced and the dispersive term is enlarged.

VI. Solitary wave in a channel

Let the elevation η be in the form of a progressive disturbance $\eta = G(\varphi)$, where $\varphi = x - Ut$. The speed of propagation U is yet unknown. We require

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[1] This derivation was not written in the original text. Introduce new variable $u = x\text{-}t$, then $P_x = P_u$, $P_t = -P_u$, the same holds for Q and η. The vanishing term in (5.25) in parenthesis next to ε: $P_t - P_x - \eta\eta_x = 0$ becomes $-2P_u = (\eta^2/2)_u$, from where P in (5.26) follows. Similarly the vanishing factor next to δ in (5.25) $Q_t - Q_x - \eta_{txx}/2 + \eta_{xxx}/6 = 0$ becomes $2Q_u = (2\eta_{xx}/3)_u$, from where Q in (5.25) follows.

that when $|\varphi| \to \infty$, $\eta \to 0$, and this holds also for $G' = G_\varphi$ and G''. The KdV (5.29) equation becomes

$$G'(c-U)+\frac{3c}{2H}GG'+\frac{cH^2}{6}G'''=0. \tag{6.1}$$

Integration of (6.1) with respect to φ gives

$$G(c-U)+\frac{3c}{4H}G^2+\frac{cH^2}{6}G''=C. \tag{6.2}$$

Since G and its derivatives have to vanish for large φ, the constant $C = 0$. We multiply (6.2) with $2G'$ and integrate again

$$\frac{cH^2}{6}G'^2+(c-U)G^2+\frac{c}{2H}G^3=0, \tag{6.3}$$

where the second integration constant was set to zero again because of the boundary conditions at infinity. Rewritten in another form (6.3) looks like

$$\left(\frac{dG}{d\varphi}\right)^2=\frac{3}{H^3}G^2(a-G), \tag{6.4}$$

where the parameter $a = 2H(U/c - 1)$ has been introduced. The solutions of the equations of the form

$$(dG/d\varphi)^2=\text{cubic in } G(\varphi) \tag{6.5}$$

are in terms of elliptic functions, like cn($\alpha\varphi$) and sn($\alpha\varphi$), where α is related to the elliptic integral, and the waves are therefore called *cnoidal waves*. The two integration constants, which we have lost in racing the solitary wave, would be nonzero, and the solution would be periodic. In our case the solution of (6.4) consists of a solitary wave (Fig. 3):

$$G=a\,\mathrm{sech}^2(b\varphi),\quad b=\sqrt{\frac{3a}{4H^3}}. \tag{6.6}$$

The solitary wave (called also the *soliton*) was first observed experimentally by John Scott Russell on the Edinburg-Glasgow Canal in 1834. The velocity of propagation is related to the amplitude of the wave:

$$U = c\left(1 + \frac{a}{2H}\right). \tag{6.7}$$

This expression for the velocity of propagation could also be obtained from the approximation of $\sqrt{g(a+H)}$ to the first order in ε = a/H. This was also discovered by Russell in laboratory experiments. Taller solitary waves travel faster than the shorter ones. Two solitons of different amplitude collide, if the taller one moves behind the shorter one (Fig. 4). It can be shown that after the collision, according to the KdV equation, their shape reappears the same as before the collision and the taller soliton moves away from the shorter one.

We can establish an appropriate extent of a soliton by looking for the value of argument $b\varphi$ at which the elevation η = a/10. This occurs when $b\varphi$ = ±1.8183. lf h = 20 m, and a = 2 m, then the wave has a length of around 133 m.

VII. Solitary wave in hydraulics

The approach which has been followed so far in obtaining the solitary wave is described in similar form by Debnath (1994), Dodd et al. (1982), and by LeBlond and Mysak (1978) and is grounded on the derivation of the KdV equation. Another, simpler approach also gives the solitary wave as a result of dynamics in shallow water and is described in textbooks of hydrodynamics (i.e. Paterson (1983) and Acheson (1990)), where the problem of hydraulic jumps (tidal bores) and shock waves is considered. Another simplified approach is followed by Defant (1961) directly from the vertical integration of Euler (2.2) and continuity (2.1) equations where the bottom boundary condition (3.4) was considered.

Paterson (1983) supposed an irrotational two-dimensional steady flow where the disturbance has been followed while it moves. The *stream function* $\psi(x, z)$ for this flow may be introduced:

$$u = \psi_z, \quad w = -\psi_x \tag{7.1}$$

The channel bottom is the streamline $\psi(x, 0) = 0$, and so is the free surface, where $\psi(x, h(x) = Q$; the constant Q is the total volume flow rate (Fig. 5). The streamline at the free surface can be expanded in a Taylor series in powers of z around the value at the bottom. The cross-channel vorticity component $\omega_y = u_z - w_x = \psi_{zz} + \psi_{xx} = 0$ for irrotational flows and we have to solve the Laplace equation for the stream function. The Taylor series of stream function, which satisfies the Laplace equation and fulfils the bottom boundary condition, follows as:

$$\psi(x,z) = zu(x,0) - \frac{z^3}{6}u_{xx}(x,0). \tag{7.2}$$

The Bernoulli equation(2.6) for a steady and irrotational flow may be written in the form:

$$p + \frac{\rho}{2}\left(u^2 + w^2\right) + \rho g z = A. \tag{7.3}$$

where A is a constant. The rate of momentum flux is balanced with the pressure force, when friction is ignored:

$$\int_{z=0}^{h} \left(p + \rho u^2\right) dz = B. \tag{7.4}$$

This integral constraint with constant B allows the elimination of pressure in (7.3), which in vertically integrated form becomes:

$$B = \int_{z=0}^{h} \left(A - \rho g z + \frac{\rho u^2}{2} - \frac{\rho w^2}{2} \right) dz. \tag{7.5}$$

The velocity components (u, w) follow from the stream function (7.2), and the integration of (7.5) gives B to third order in h:

$$B \cong Ah - \frac{\rho g h^2}{2} + \frac{\rho}{2}\left(u^2 h - \frac{h^3 u u''}{3} \right) - \frac{\rho h^3 u'^2}{6} \tag{7.6}$$

The velocity u is the velocity at the channel bottom. Since the stream function (7.2) is constant (Q) at the surface, the slip velocity along the bottom can be expressed in terms of Q. The approximation $u(x,\ 0) \cong Q/h(x)$ is sufficient to obtain:

$$B \cong Ah - \frac{\rho g h^2}{2} + \frac{\rho Q^2}{2h} - \frac{\rho Q^2 (h')^2}{6h}. \tag{7.7}$$

This is the differential equation for the height of the water column in a channel which could be rewritten in the form of (6.5), which gives the cnoidal and solitary waves.

The left-hand side of (6.4) implies that the right-hand side has to be positive or zero. Since from (6.6) it follows that $G \geq a$, a has to be positive (a "negative" hump cannot be a solution at the fluid surface). From (6.7) it follows that this is achieved if:

$$F = \frac{U}{c} > 1. \tag{7.8}$$

The solitary wave happens only in a supercritical flow. In a laboratory, a soliton can be produced by moving a piston along an elongated container. If the piston, however, moves too fast, the resulting length of disturbance is too short and the amplitude is high, which leads to the hydraulic jump (bore) rather than a solitary wave. This happens when $\varepsilon/\delta = aL/H^3 > 16$. The non-linear steepening wins against the dispersion. It has to be mentioned that it is not necessary that all of the kinetic energy of a disturbance is transformed into a random local motion on a small scale, but that also a packet of cnoidal waves may emanate from the bore, forming an "undular" bore.

VIII. Conclusion

These lecture notes are meant to be an introduction into the field of nonlinear waves. For this reason only the surface waves in a flat-bottomed channel were considered. The solitary waves were also found in a stratified fluid as internal nonlinear waves. The flow is no more irrotational ($\nabla\rho \times \nabla p/\rho^2$ is the rate of vorticity changes), and the conservation of density $D\rho/Dt = 0$ of fluid particle has to be considered, if neither compression nor diffusion takes place. Nonlinear internal waves are described in LeBlond and Mysak (1978). The theory simplifies when the rigid lid is presumed to be at the free surface and the stratification is simple (Fig. 6). In a two layer fluid a solitary wave at an interface of two fluids may be one of elevation or depression. Let h_1 and ρ_1 denote the thickness and density of an upper layer, and h_2 and ρ_2 are the thickness and density of a lower layer. If $(h_1/h_2)^2 > \rho_1/\rho_2$, then the solitary wave will be one of elevation, while if the opposite happens, the soliton will be one of depression. If $(h_1/h_2)^2 = \rho_1/\rho_2$, then no solitary wave may be generated. The conditions for these cases were discussed by Thorpe (1968).

FIGURES

Fig. l. The sketch of wave geometry in a channel.

Fig.2. The linear solution of waves. a) Deep water, b) Intermedium depth, c) Shallow water.

Fig.3. A solitary wave.

Fig.4. Collision of two solitary waves. A taller solitary wave catches up the shorter one, they interact nonlinearly according to the KdV equation, and the taller soliton passes away from the shorter one.

Fig. 5. Sketch for the hydraulic approach to solitary wave in a channel.

Fig. 6. Forms of internal waves (redrawn after Thorpe, 1968).

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FIG. 1

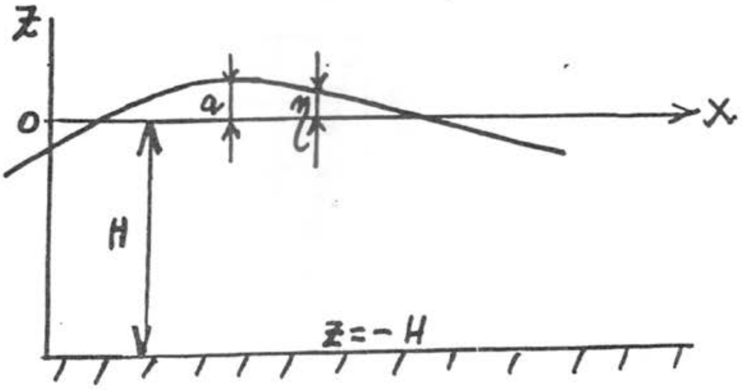

FIG. 2

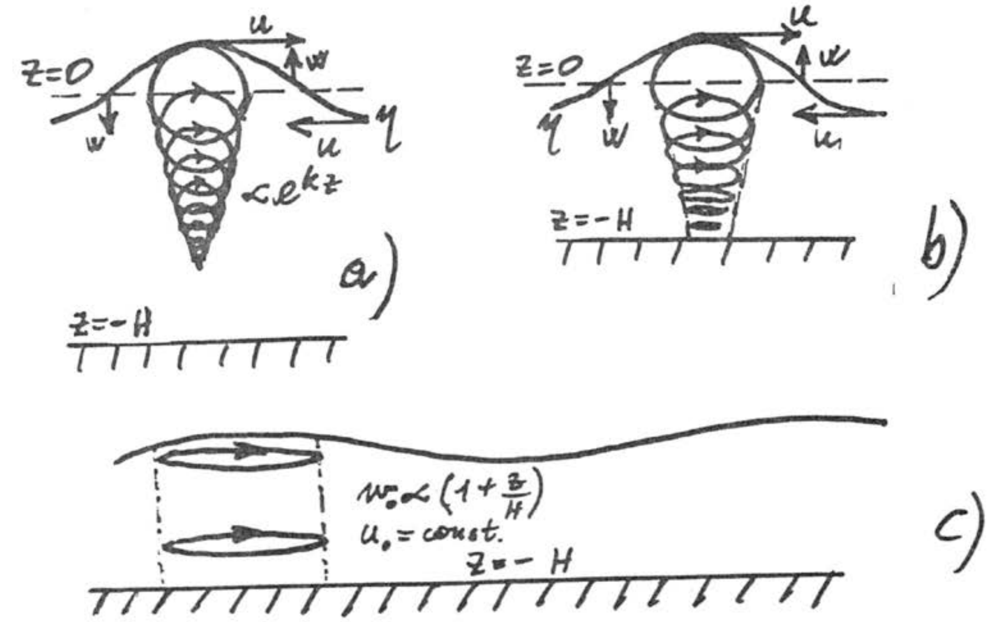

FIG. 3

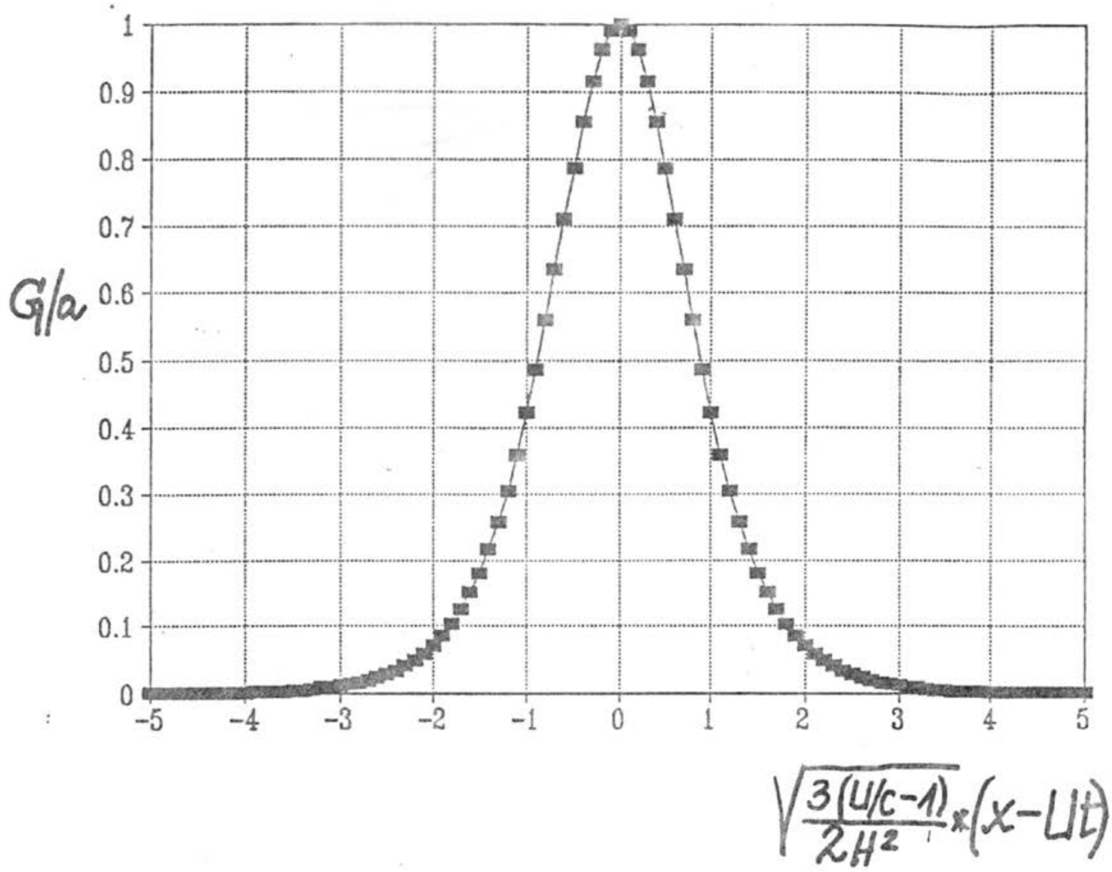
G/a
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−5
−4
−3
−2
−1
0
1
2
3
4
5
$\sqrt{\frac{3(U/c-1)}{2H^2}}*(x-Ut)$

FIG. 4

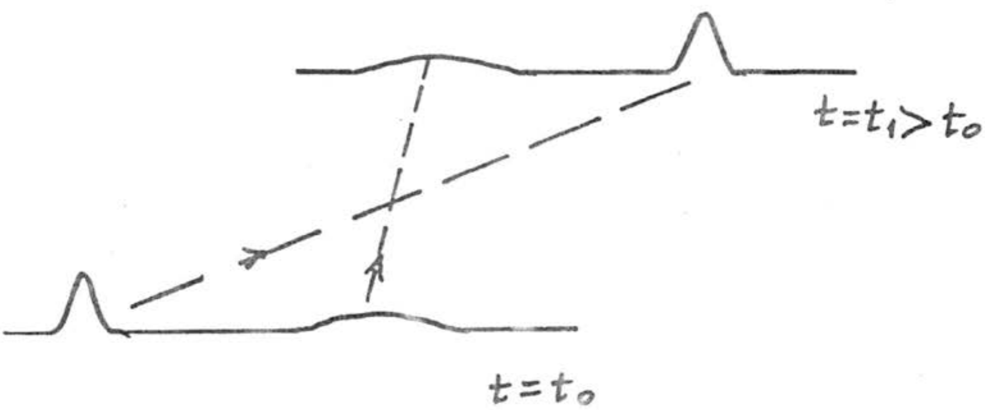
$t=t_1>t_0$
$t=t_0$

FIG. 5

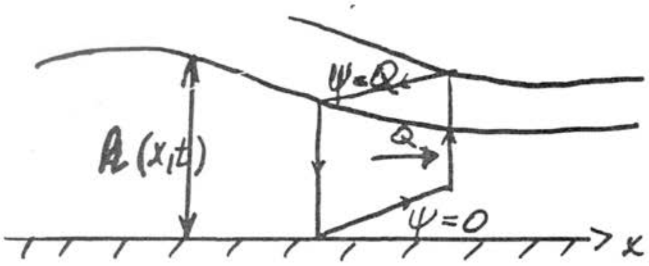

FIG. 6